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LETTER TO THE EDITOR

The noncommutative degenerate electron gas

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Abstract

The quantum dynamics of nonrelativistic single-particle systems involving noncommutative coordinates, usually referred to as noncommutative quantum mechanics, has lately been the object of several investigations. In this letter we pursue these studies for the case of multi-particle systems. We use as a prototype the degenerate electron gas whose dynamics is well known in the commutative limit. Our central aim here is to understand qualitatively, rather than quantitatively, the main modifications induced by the presence of noncommutative coordinates. We shall first see that the noncommutativity modifies the exchange correlation energy while preserving the electric neutrality of the model. By employing time-independent perturbation theory together with the Seiberg–Witten map we show, afterwards, that the ionization potential is modified by the noncommutativity. It also turns out that the noncommutative parameter acts as a reference temperature. Hence, the noncommutativity lifts the degeneracy of the zero temperature electron gas.

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The first paper on quantum field theories formulated in a noncommutative spacetime manifold was published in 1947 [1], although the idea that a noncommutative spacetime manifold might provide a solution for the problem of ultraviolet divergences seems to have been suggested long before [2]. The subject was, perhaps, abandoned due to the success of renormalization theory and its revival is rather recent and related to string theory. Indeed, the noncommutative Yang–Mills theory arises as a limit of string theory [3] and it was extracted by Seiberg and Witten [4] by starting from the open string in the presence of a magnetic field. More details on this and related subjects can be found in the already existing review articles [5–10] and also in the specialized literature.

On the other hand, noncommutative quantum mechanics has also been under scrutiny [11–13]. The main outcome, in the case of single-particle systems, is that a modification of the equal-time algebra obeyed by the basic position observables acts as a source of new interactions which may or may not preserve the original symmetries. This paper is dedicated to study the physical consequences of introducing noncommutative coordinates in the case of quantum many-particle systems.

We consider, as a prototype, an idealized high density degenerate electron gas occupying a volume $V = L^3$. As is currently assumed, the electrons (electric charge $-e$) are in the presence of a uniform background of positive ions (electric charge $+e$) that makes the whole system electrically neutral. For this to be the case the number of electrons must equal the number of ions (N). The ions are much heavier than the electrons and will, then, be considered as static. Although this system has been extensively described in textbooks [14] we make a small digression here to pinpoint its highlights. By election, the degrees of freedom of each electron (E) are the Cartesian positions $\{X^j_{a,E}\}$ and linear momenta $\{P^j_{a,E}\}$, together with the spins $\{S^j_{a,E}\}$. For the ions (B), the corresponding observables are, respectively, $\{X^j_{a,B}\}$, $\{P^j_{a,B}\}$ and $\{S^j_{a,B}\}$. Lower case letters from the beginning of the Latin alphabet $(a, b, \ldots)$ designate the particle, while lower case letters from the middle of the Latin alphabet $(i, j, \ldots)$ only run from 1 to 3 and identify the Cartesian component of the corresponding observable. Observables associated with the electrons commute with those associated with the ions. The phase space electron degrees of freedom obey the standard equal-time algebra

$$\left[X^i_{a,E}, X^j_{b,E}\right] = 0, \qquad \left[X^i_{a,E}, P^j_{b,E}\right] = \mathrm{i}\hbar\delta^{ij}\delta_{ab}, \qquad \left[P^i_{a,E}, P^j_{b,E}\right] = 0. \tag{1}$$

We emphasize that all position observables commute among themselves. The equal-time algebra for the ion phase space variables can be obtained from equation (1) just by replacing E by B. The algebra of the spin components will not be explicitly displayed.

Structurally, the more general form of the total Hamiltonian reads

$$H = H_E + H_B + H_{EB}. \tag{2}$$

The Hamiltonian H_E describes the free dynamics of the electrons plus the Coulomb interaction among them. Hence, in the position representation ($X^i\,|\vec{x}\rangle = x^i\,|\vec{x}\rangle$, $P^i \to p^i = -\mathrm{i}\hbar\partial/\partial x^i$),

$$H_E = \sum_{a=1}^{N} \frac{p^i_{a,E}\, p^i_{a,E}}{2m} + \frac{1}{2}\sum_{a\neq b}^{N} \mathrm{V}(|\vec{x}_{a,E} - \vec{x}_{b,E}|), \tag{3}$$

where

$$\mathrm{V}(|\vec{r}|) = e^2 \frac{\mathrm{e}^{-\mu|\vec{r}|}}{|\vec{r}|} \tag{4}$$

and μ is a damping factor needed to secure, in the thermodynamic limit, the convergence of each term in the right-hand side of equation (2) [14]. As for H_B one writes

$$\begin{aligned} H_B &= \frac{1}{2}\sum_{a\neq b}^{N} \mathrm{V}(|\vec{x}_{a,B} - \vec{x}_{b,B}|) \\ &\to \frac{1}{2}\int \mathrm{d}^3x_{a,B} \int \mathrm{d}^3x_{b,B}\, n(\vec{x}_{a,B})\, n(\vec{x}_{b,B})\, \mathrm{V}(|\vec{x}_{a,B} - \vec{x}_{b,B}|). \end{aligned} \tag{5}$$

The absence of a kinetic term in equation (5) reflects the fact that the ions are static. Furthermore, the continuous nature of the ion background is taken into account by replacing the discrete summations by continuous integrals. This brings into play the new variable,

$n(\vec{x}_{a,B})$, known as the ion density. Finally, H_{EB} is taken to be

$$H_{EB} = -\sum_{a=1}^{N}\sum_{b=1}^{N} \mathrm{V}(|\vec{x}_{a,E} - \vec{x}_{b,B}|)$$
$$\rightarrow -\sum_{a=1}^{N}\int \mathrm{d}^3 x_{b,B}\, n(\vec{x}_{b,B})\mathrm{V}(|\vec{x}_{a,B} - \vec{x}_{b,B}|). \tag{6}$$

We shall always be working in the approximation $n(\vec{x}_{a,B}) = \text{constant} = N/V$.

It has long been shown [14] that H, in equation (2), can be cast

$$H = H_0 + H_I, \tag{7}$$

where

$$H_0 = \sum_{\vec{k}\lambda} \frac{\hbar^2 k^2}{2m} c^{\dagger}_{\vec{k}\lambda} c_{\vec{k}\lambda}, \tag{8a}$$

$$H_I = \frac{2\pi}{V} \sum_{\vec{k}\vec{p}\vec{q}}{}' \sum_{\lambda_1\lambda_2} \frac{e^2}{q^2} c^{\dagger}_{\vec{k}+\vec{q},\lambda_1} c^{\dagger}_{\vec{p}-\vec{q},\lambda_2} c_{\vec{p},\lambda_2} c_{\vec{k},\lambda_1}. \tag{8b}$$

Here, $c^{\dagger}_{\vec{k}\lambda}$ $(c_{\vec{k}\lambda})$ are the creation (annihilation) operators of electrons of momentum $\vec{k}$ and spin λ, whereas $q \equiv |\vec{q}|$. Furthermore, the prime in the summation symbol implies that the momentum $\vec{q} = 0$ is excluded. Within the framework of time-independent perturbation theory, the main outcomes, including contributions up to the second order, may be summarized as follows. The unperturbed ground state (Fermi) energy $\left(E_0^{(0)}\right)$ is given by [14]

$$E_0^{(0)} = \frac{e^2}{2a_0} N \frac{2.21}{r_s^2}, \tag{9}$$

where a_0 is the Bohr radius, $r_s \equiv r_0/a_0$ and $\frac{4}{3}\pi r_0^3 = V/N$. Moreover, its first- and second-order perturbative corrections were, respectively, found to read [14–17]

$$E_0^{(1)} = -\frac{e^2}{2a_0} N \frac{0.916}{r_s}, \tag{10}$$

and

$$E_0^{(2)} = \frac{e^2}{2a_0} N\left[\epsilon_0^{(2)r} + \epsilon_0^{(2)b}\right] = \frac{e^2}{2a_0} N\,[0.0622 \ln r_s - 0.094]\,. \tag{11}$$

Here [18],

$$\epsilon_0^{(2)b} = \frac{3}{16\pi^5} \int \frac{\mathrm{d}^3\mathbf{q}}{\mathbf{q}^2} \int_{|\mathbf{k}+\mathbf{q}|>1} \mathrm{d}^3\mathbf{k} \int_{|\mathbf{p}-\mathbf{q}|>1} \mathrm{d}^3\mathbf{p} \frac{\xi(1-\mathbf{k})\xi(1-\mathbf{p})}{[\mathbf{q}\cdot(\mathbf{q}+\mathbf{k}-\mathbf{p})](\mathbf{q}+\mathbf{k}-\mathbf{p})^2}$$
$$= \frac{1}{3}\ln 2 - \frac{3}{2\pi^2}\zeta(3) \approx 0.048, \tag{12}$$

is the exchange correlation energy in Rydberg units. We shall designate by $\xi(x)$ the Heaviside step function. In order to work with dimensionless vectors ($\vec{\mathbf{p}}$) we define $k_F\,\vec{\mathbf{p}} \equiv \vec{p}$, where $k_F \equiv \left(\frac{9\pi}{4}\right)^{\frac{1}{3}} r_0^{-1}$ is the Fermi wavenumber. Also, $\mathbf{p} \equiv |\vec{\mathbf{p}}|$. Here ends our brief summary about the degenerate electron gas.

We turn next to studying the implications of replacing $X^i_{a,E} \rightarrow Q^i_{a,E}$, $P^i_{a,E} \rightarrow P^i_{a,E}$, now obeying the equal-time phase space algebra

$$\left[Q^i_{a,E}, Q^j_{b,E}\right] = 2\mathrm{i}\delta_{a,b}\Theta^{ij}_E, \qquad \left[Q^i_{a,E}, P^j_{b,E}\right] = \mathrm{i}\hbar\delta^{ij}\delta_{ab}, \qquad \left[P^i_{a,E}, P^j_{b,E}\right] = 0. \tag{13}$$

The distinctive feature of the new position observables $\left(Q^{i}_{a,E}\right)$ is that they do not commute among themselves. This lack of noncommutativity is characterized by the real antisymmetric constant matrix $\left(\Theta^{ij}_{E}\right)$. An explicit representation for this algebra has already been obtained [10–13] after realizing that (see equations (1) and (14))

$$Q^{i}_{a,E} = X^{i}_{a,E} - \frac{1}{\hbar}\Theta^{ij}_{E}P^{j}_{a,E}. \tag{14}$$

A similar modification should be introduced for the ions. However, for static ions equation (14) reduces to $Q^{i}_{a,B} = X^{i}_{a,B}$.

As for the Hamiltonians, the replacement $X^{i}_{a,E} \to Q^{i}_{a,E}$, $P^{i}_{a,E} \to P^{i}_{a,E}$ amounts to $H \to \mathcal{H}$, such that

$$\mathcal{H} = \mathcal{H}_{E} + \mathcal{H}_{B} + \mathcal{H}_{EB}, \tag{15}$$

where

$$\mathcal{H}_{E} = \sum_{a=1}^{N}\frac{p^{i}_{a,E}p^{i}_{a,E}}{2m} + \frac{1}{2}\sum_{a\neq b}^{N}\mathrm{V}(|\vec{\phi}_{a,E} - \vec{\phi}_{b,E}|), \tag{16}$$

$$\begin{aligned}\mathcal{H}_{B} &= \frac{1}{2}\int \mathrm{d}^{3}x_{a,B}\int \mathrm{d}^{3}x_{b,B}\, n(\vec{x}_{a,B})n(\vec{x}_{b,B})\mathrm{V}(|\vec{x}_{a,B} - \vec{x}_{b,B}|)\\ &= H_{B}\end{aligned} \tag{17}$$

and

$$\mathcal{H}_{EB} = -\sum_{a=1}^{N}\int \mathrm{d}^{3}x_{b,B}\, n(\vec{x}_{b,B})\mathrm{V}(|\vec{\phi}_{a,E} - \vec{x}_{b,B}|). \tag{18}$$

For simplifying purposes, we have introduced the notation

$$\phi^{i}_{a,E} \equiv x^{i}_{a,E} - \frac{1}{\hbar}\Theta^{ij}_{E}p^{i}_{a,E}. \tag{19}$$

As already stated, $\left\{x^{i}_{a,E}\right\}$ denotes the set of eigenvalues of the operator $X^{i}_{a,E}$, whereas $p^{i}_{a,E} \equiv -\mathrm{i}\hbar\partial/\partial x^{i}_{a,E}$ represents $P^{i}_{a,E}$ in the basis defined by the common eigenvectors of $\left\{X^{i}_{a,E}\right\}$.

We next focus on $\mathrm{V}(|\vec{\phi}_{a,E} - \vec{x}_{b,B}|)$ when acting on an arbitrary but differentiable function $\Psi\left(x^{i}_{a,E}, x^{i}_{b,B}\right)$. By taking into account equation (19) one finds that

$$\begin{aligned}&\mathrm{V}\left(\left|x^{i}_{a,E} - \frac{1}{\hbar}\Theta^{ij}_{E}p^{j}_{a,E} - x^{i}_{b,B}\right|\right)\Psi\left(x^{i}_{a,E}, x^{i}_{b,B}\right)\\ &\quad= \frac{1}{(2\pi)^{3/2}}\int \mathrm{d}^{3}k\,\tilde{\mathrm{V}}(\vec{k})\exp\left\{\mathrm{i}k^{i}\left(x^{i}_{a,E} - \frac{1}{\hbar}\Theta^{ij}_{E}p^{j}_{a,E} - x^{i}_{b,B}\right)\right\}\Psi\left(x^{i}_{a,E}, x^{i}_{b,B}\right)\\ &\quad= \frac{1}{(2\pi)^{3/2}}\int \mathrm{d}^{3}k\,\tilde{\mathrm{V}}(\vec{k})\exp\{\mathrm{i}\vec{k}\cdot(\vec{x}_{a,E} - \vec{x}_{b,B})\}\exp\left\{-k^{i}\Theta^{ij}_{E}\partial^{j}_{\vec{x}_{a,E}}\right\}\Psi\left(x^{i}_{a,E}, x^{i}_{b,B}\right)\\ &\quad= \mathrm{V}\left(\left|x^{i}_{a,E} - x^{i}_{b,B}\right|\right)\exp\left\{\mathrm{i}\,\overleftarrow{\partial^{i}}_{\vec{x}_{a,E}}\Theta^{ij}_{E}\overrightarrow{\partial^{j}}_{\vec{x}_{a,E}}\right\}\Psi\left(x^{i}_{a,E}, x^{i}_{b,B}\right)\\ &\quad= \mathrm{V}\left(\left|x^{i}_{a,E} - x^{i}_{b,B}\right|\right)\star_{a,E}\Psi\left(x^{i}_{a,E}, x^{i}_{b,B}\right),\end{aligned} \tag{20}$$

where $\tilde{\mathrm{V}}(\vec{k})$ is the Fourier transform of $\mathrm{V}(|\vec{x}|)$ and

$$\mathrm{V}(|\vec{x}_{a,E} - \vec{x}_{b,B}|)\star_{a,E}\Psi\left(x^{i}_{a,E}, x^{i}_{b,B}\right) \equiv \mathrm{V}(|\vec{x}_{a,E} - \vec{x}_{b,B}|)\,\mathrm{e}^{\mathrm{i}\,\overleftarrow{\partial^{i}}_{\vec{x}_{a,E}}\Theta^{ij}\overrightarrow{\partial^{j}}_{\vec{x}_{a,E}}}\Psi\left(x^{i}_{a,E}, x^{i}_{b,B}\right) \tag{21}$$

is the Grönewold–Moyal or $\star$-product [19, 20]. Note that $x^{i}_{a,E}$ does not commute with $p^{i}_{a,E}$ but, however, it does commute with $\Theta^{ij}_{E}p^{j}_{a,E}$ due to the antisymmetric character of Θ^{ij}_{E}. This observation is of the outmost importance for arriving at equation (20). Furthermore, since only the electron coordinates are sensitive to the $\star$-product we drop, from now on, the subscript E in this particular symbol. We single out

$$\int \mathrm{d}^3x\, \phi_1(\mathbf{x}) \star \phi_2(\mathbf{x}) = \int \mathrm{d}^3x\, \phi_1(\mathbf{x})\phi_2(\mathbf{x}), \tag{22a}$$

$$\begin{aligned}\int \mathrm{d}^3x\, \phi_1(\mathbf{x}) \star \phi_2(\mathbf{x}) \star \phi_3(\mathbf{x}) &= \int \mathrm{d}^3x\, \phi_3(\mathbf{x}) \star \phi_1(\mathbf{x}) \star \phi_2(\mathbf{x}) \\ &= \int \mathrm{d}^3x\, \phi_2(\mathbf{x}) \star \phi_3(\mathbf{x}) \star \phi_1(\mathbf{x}),\end{aligned} \tag{22b}$$

as the properties of the $\star$-product [6–9] which will play a relevant role in our future developments.

We now address the problem of computing $\mathcal{H}_{EB}$. By substituting equation (20) into equation (18) one obtains

$$\begin{aligned}\mathcal{H}_{EB}\Psi\left(x^i_{c,E}, x^i_{d,B}\right) &= -\sum_{a=1}^{N}\int \mathrm{d}^3x_{b,B}\, n(\vec{x}_{b,B})\mathrm{V}\left(\left|x^i_{a,E} - \frac{1}{\hbar}\Theta^{ij}_E p^j_{a,E} - x^i_{b,B}\right|\right)\Psi\left(x^i_{c,E}, x^i_{d,B}\right) \\ &= -\frac{N}{V}\sum_{a=1}^{N}\int \mathrm{d}^3x_{b,B}\,\mathrm{V}\left(\left|x^i_{a,E} - x^i_{b,B}\right|\right) \star_a \Psi\left(x^i_{c,E}, x^i_{d,B}\right) \\ &= -\frac{N}{V}\sum_{a=1}^{N}\left[\int \mathrm{d}^3z\,\mathrm{V}(|\vec{z}|)\right] \star_a \Psi\left(x^i_{c,E}, x^i_{d,B}\right) \\ &= H_{EB}\Psi\left(x^i_{c,E}, x^i_{d,B}\right),\end{aligned} \tag{23}$$

which in view of the arbitrariness of $\Psi\left(x^i_{c,E}, x^i_{d,B}\right)$ amounts to

$$\mathcal{H}_{EB} = H_{EB}, \tag{24}$$

as an operator identity. Thus, the noncommutativity of the electron position observables does not affect the Hamiltonian H_{EB}. This is a consequence of the continuous structure assumed for the ion background.

It remains to study the modifications induced by the noncommutativity on H_E. One may convince oneself that

$$\mathcal{H}_E = H_0 + \mathcal{V}_E, \tag{25}$$

where H_0 is given in equation (8a), while

$$\mathcal{V}_E = \frac{1}{2}\sum_{\vec{k}_1\lambda_1}\sum_{\vec{k}_2\lambda_2}\sum_{\vec{k}_3\lambda_3}\sum_{\vec{k}_4\lambda_4} c^\dagger_{\vec{k}_1\lambda_1} c^\dagger_{\vec{k}_2\lambda_2}\langle\vec{k}_1\lambda_1\vec{k}_2\lambda_2|\mathrm{V}(|\vec{Q}_{a,E} - \vec{Q}_{b,E}|)|\vec{k}_3\lambda_3\vec{k}_4\lambda_4\rangle c_{\vec{k}_4\lambda_4} c_{\vec{k}_3\lambda_3}. \tag{26}$$

Through standard manipulations [14], the right-hand side of equation (26) can be written as

$$\begin{aligned}&\langle\vec{k}_1\lambda_1\vec{k}_2\lambda_2|\mathrm{V}(|\vec{Q}_{a,E} - \vec{Q}_{b,E}|)|\vec{k}_3\lambda_3\vec{k}_4\lambda_4\rangle \\ &\quad= V^{-2}\left[\eta^\dagger_{\lambda_1}(a)\otimes\eta^\dagger_{\lambda_2}(b)\right]\left[\eta_{\lambda_3}(a)\otimes\eta_{\lambda_4}(b)\right]\int \mathrm{d}^3x_{a,E}\int \mathrm{d}^3x_{b,E}\, \mathrm{e}^{-\mathrm{i}\vec{k}_1\cdot\vec{x}_{a,E}} \\ &\qquad\times \mathrm{e}^{-\mathrm{i}\vec{k}_2\cdot\vec{x}_{b,E}}\left[\mathrm{V}(|\vec{x}_{a,E} - \vec{x}_{b,E}|) \star_a \star_b \mathrm{e}^{\mathrm{i}\vec{k}_3\cdot\vec{x}_{a,E}}\,\mathrm{e}^{\mathrm{i}\vec{k}_4\cdot\vec{x}_{b,E}}\right],\end{aligned} \tag{27}$$

where

$$\Phi_{\vec{k}\lambda}(\vec{x}) \equiv \langle\vec{x}|\vec{k}\lambda\rangle = V^{-\frac{1}{2}}\,\mathrm{e}^{\mathrm{i}\vec{x}\cdot\vec{k}}\eta_\lambda \tag{28}$$

is the free electron wavefunction, with

$$\eta_\uparrow = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \eta_\downarrow = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{29}$$

Furthermore,

$$k_i = \frac{2\pi n_i}{L}, \qquad i = 1, 2, 3 \quad \text{and} \quad n_i = \pm 1, \pm 2, \ldots, \tag{30}$$

is the periodically quantized momentum. The use of equations (22*a*) and (22*b*) enables us to find

$$\begin{aligned}
&\langle \vec{k}_1 \lambda_1 \vec{k}_2 \lambda_2 | \mathrm{V}(|\vec{Q}_{a,E} - \vec{Q}_{b,E}|) | \vec{k}_3 \lambda_3 \vec{k}_4 \lambda_4 \rangle \\
&\quad = V^{-2} \delta_{\lambda_1 \lambda_3} \delta_{\lambda_2 \lambda_4} \int \mathrm{d}^3 x_{a,E} \int \mathrm{d}^3 x_{b,E}\, \mathrm{V}(|\vec{x}_{a,E} - \vec{x}_{b,E}|) \\
&\qquad \times [\mathrm{e}^{\mathrm{i}\vec{k}_3 \cdot \vec{x}_{a,E}} \star_a \mathrm{e}^{-\mathrm{i}\vec{k}_1 \cdot \vec{x}_{a,E}}][\mathrm{e}^{\mathrm{i}\vec{k}_4 \cdot \vec{x}_{b,E}} \star_b \mathrm{e}^{-\mathrm{i}\vec{k}_2 \cdot \vec{x}_{b,E}}] \\
&\quad = V^{-1} \delta_{\vec{k}_1 + \vec{k}_2, \vec{k}_3 + \vec{k}_4} \delta_{\lambda_1 \lambda_3} \delta_{\lambda_2 \lambda_4}\, \mathrm{e}^{\mathrm{i}(\vec{k}_3 \wedge \vec{k}_1 + \vec{k}_4 \wedge \vec{k}_2)} \int \mathrm{d}^3 z\, \mathrm{V}(|\vec{z}|)\, \mathrm{e}^{\mathrm{i}\vec{z} \cdot (\vec{k}_4 - \vec{k}_2)},
\end{aligned} \tag{31}$$

where the wedge product stands for

$$\vec{k} \wedge \vec{p} \equiv k^i \Theta_E^{ij} p^j. \tag{32}$$

To arrive at the last term in the right-hand side of equation (31), we took advantage of

$$\mathrm{e}^{\mathrm{i}\vec{x} \cdot \vec{k}} \star \mathrm{e}^{-\mathrm{i}\vec{x} \cdot \vec{p}} = \mathrm{e}^{\mathrm{i}\vec{k} \wedge \vec{p}}\, \mathrm{e}^{\mathrm{i}\vec{x} \cdot \vec{k}}\, \mathrm{e}^{-\mathrm{i}\vec{x} \cdot \vec{p}}. \tag{33}$$

The substitution of equation (31) into equation (26) yields

$$\begin{aligned}
\mathcal{V}_E &= \frac{1}{2} \sum_{\vec{k}_1 \lambda_1} \sum_{\vec{k}_2 \lambda_2} \sum_{\vec{k}_3 \lambda_3} \sum_{\vec{k}_4 \lambda_4} c^\dagger_{\vec{k}_1 \lambda_1} c^\dagger_{\vec{k}_2 \lambda_2} V^{-1} \delta_{\vec{k}_1 + \vec{k}_2, \vec{k}_3 + \vec{k}_4} \delta_{\lambda_1 \lambda_3} \delta_{\lambda_2 \lambda_4} \mathrm{e}^{\mathrm{i}(\vec{k}_3 \wedge \vec{k}_1 + \vec{k}_4 \wedge \vec{k}_2)} \\
&\qquad \times \int \mathrm{d}^3 z\, \mathrm{V}(|\vec{z}|)\, \mathrm{e}^{\mathrm{i}\vec{z} \cdot (\vec{k}_4 - \vec{k}_2)} c_{\vec{k}_4 \lambda_4} c_{\vec{k}_3 \lambda_3} \\
&= \frac{1}{2} V^{-1} \sum_{\vec{k}\vec{p}\vec{q}} \sum_{\lambda_1 \lambda_2} \left[\int \mathrm{d}^3 z\, \mathrm{V}(|\vec{z}|)\, \mathrm{e}^{\mathrm{i}\vec{q} \cdot \vec{z}} \right] \mathrm{e}^{-\mathrm{i}\vec{q} \wedge (\vec{k} - \vec{p})} c^\dagger_{\vec{k} + \vec{q}, \lambda_1} c^\dagger_{\vec{p} - \vec{q}, \lambda_2} c_{\vec{p} \lambda_2} c_{\vec{k} \lambda_1}.
\end{aligned} \tag{34}$$

This is the desired form of $\mathcal{V}_E$ in terms of creation and annihilation operators. It exhibits explicitly the noncommutativity. As is common practice, we have chosen $\vec{q}$ to designate the momentum transfer of the reaction $\vec{p} + \vec{k} \to (\vec{p} - \vec{q}) + (\vec{k} + \vec{q})$.

As in the commutative case [14], the contribution of the $\vec{q} = 0$ mode in the right-hand side of equation (34) cancels out those arising from $\mathcal{H}_B$ and $\mathcal{H}_{EB}$. This means that the noncommutativity does not destroy the electric neutrality. Hence, the whole modified system collapses into

$$\mathcal{H} = H_0 + \mathcal{H}_I, \tag{35}$$

where H_0 is given by equation (8*a*), while $\mathcal{H}_I$ reads

$$\mathcal{H}_I = \frac{2\pi e^2}{V} \sum_{\vec{k}\vec{p}\vec{q}}{}' \sum_{\lambda_1 \lambda_2} \frac{\mathrm{e}^{-\mathrm{i}\vec{q} \wedge (\vec{k} - \vec{p})}}{q^2} c^\dagger_{\vec{k} + \vec{q}, \lambda_1} c^\dagger_{\vec{p} - \vec{q}, \lambda_2} c_{\vec{p}, \lambda_2} c_{\vec{k}, \lambda_1}. \tag{36}$$

At this point a digression is in order. Note that, in contradistinction to the relativistic case, the commutative limit $\left(\Theta_E^{ij} \to 0\right)$ in equation (36) exists and is well defined. To put it differently, the UV/IR mechanism [21], that contaminates noncommutative relativistic field theories, does not presently arise. This is of course due to the absence of ultraviolet divergences in the nonrelativistic case. It is a rather simple exercise to verify that $\mathcal{H}_I$ is Hermitian, as it must be.

We have so far developed the tools to compute some of the physical effects induced by the noncommutativity in the electron gas. We focus on the ground state energy eigenvalue and employ, as in the commutative situation, time-independent perturbation theory. We start by writing

$$\mathcal{E}_0 = E_0^{(0)} + \langle E_0^{(0)} | \mathcal{H}_I | E_0^{(0)} \rangle + \sum_{i \neq 0} \frac{|\langle E_0^{(0)} | \mathcal{H}_I | E_i^{(0)} \rangle|^2}{E_0^{(0)} - E_i^{(0)}} + \cdots, \tag{37}$$

where $\{E_i^{(0)}\}$ are the excited states of H_0. Since H_0 does not feel the presence of noncommutativity, its eigenstates and corresponding eigenvalues remain unchanged. Therefore, equation (9) still holds true.

What come next is the computation of $\mathcal{E}_0^{(1)}$ which, according to equation (37), reads

$$\begin{aligned}\mathcal{E}_0^{(1)} &= \langle E_0^{(0)} | \mathcal{H}_I | E_0^{(0)} \rangle \\ &= \frac{2\pi e^2}{V} \sum_{\vec{k}\vec{p}\vec{q}}{}' \sum_{\lambda_1\lambda_2} \frac{\mathrm{e}^{-\mathrm{i}\vec{q}\wedge(\vec{k}-\vec{p})}}{q^2} \langle E_0^{(0)} | c^\dagger_{\vec{k}+\vec{q},\lambda_1} c^\dagger_{\vec{p}-\vec{q},\lambda_2} c_{\vec{p},\lambda_2} c_{\vec{k},\lambda_1} | E_0^{(0)} \rangle.\end{aligned} \tag{38}$$

As already indicated, the mode $\vec{q} = 0$ does not contribute to the right-hand side of equation (38). Then, straightforward manipulations lead us to

$$\langle E_0^{(0)} | c^\dagger_{\vec{k}+\vec{q},\lambda_1} c^\dagger_{\vec{p}-\vec{q},\lambda_2} c_{\vec{p},\lambda_2} c_{\vec{k},\lambda_1} | E_0^{(0)} \rangle = -\xi(k_F - p)\xi(k_F - k)\delta_{\vec{p}-\vec{q},\vec{k}}\delta_{\lambda_1\lambda_2}. \tag{39}$$

Observe now that, when substituting equation (39) into equation (38), the factor $\delta_{\vec{p}-\vec{q},\vec{k}}$ kills all noncommutative effects and, therefore

$$\mathcal{E}_0^{(1)} = E_0^{(1)}. \tag{40}$$

The computation of $\mathcal{E}_0^{(2)}$,

$$\mathcal{E}_0^{(2)} = \sum_{i \neq 0} \frac{|\langle E_0^{(0)} | \mathcal{H}_I | E_i^{(0)} \rangle|^2}{E_0^{(0)} - E_i^{(0)}}, \tag{41}$$

is cumbersome. We shall not pause here to present the details but merely mention that it turns out to be given by

$$\mathcal{E}_0^{(2)} = \frac{e^2}{2a_0} N \left[\epsilon_0^{(2)r} + \epsilon_0^{(2)b}(\Theta)\right]. \tag{42}$$

It is instructive to compare this result with its commutative counterpart, quoted in equations (11) and (12). On one hand, $\epsilon_0^{(2)r}$ remains unaffected by the noncommutativity while, on the other hand, the exchange correlation energy term, $\epsilon_0^{(2)b}$, is modified as follows

$$\epsilon_0^{(2)b} \to \epsilon_0^{(2)b}(\Theta) = \frac{3}{16\pi^5} \int \frac{\mathrm{d}^3\mathbf{q}}{\mathbf{q}^2} \int_{|\vec{\mathbf{k}}+\vec{\mathbf{q}}|>1} \mathrm{d}^3\mathbf{k} \int_{|\vec{\mathbf{p}}-\vec{\mathbf{q}}|>1} \mathrm{d}^3\mathbf{p} \frac{\xi(1-\mathbf{k})\xi(1-\mathbf{p})\,\mathrm{e}^{-2\mathrm{i}k_F^2\vec{\mathbf{q}}\wedge(\vec{\mathbf{k}}-\vec{\mathbf{p}})}}{[\vec{\mathbf{q}}\cdot(\vec{\mathbf{q}}+\vec{\mathbf{k}}-\vec{\mathbf{p}})](\vec{\mathbf{q}}+\vec{\mathbf{k}}-\vec{\mathbf{p}})^2}. \tag{43}$$

One may easily verify that $\epsilon_0^{(2)b}(\Theta)$ is real, as demanded by the Hermiticity of $\mathcal{H}_I$. This allows the replacement of the exponential by its real part, i.e.,

$$\epsilon_0^{(2)b}(\Theta) = \frac{3}{16\pi^5} \int \frac{\mathrm{d}^3\mathbf{q}}{\mathbf{q}^2} \int_{|\vec{\mathbf{k}}+\vec{\mathbf{q}}|>1} \mathrm{d}^3\mathbf{k} \int_{|\vec{\mathbf{p}}-\vec{\mathbf{q}}|>1} \mathrm{d}^3\mathbf{p} \frac{\xi(1-\mathbf{k})\xi(1-\mathbf{p})\cos\left[2k_F^2\vec{\mathbf{q}}\wedge(\vec{\mathbf{k}}-\vec{\mathbf{p}})\right]}{[\vec{\mathbf{q}}\cdot(\vec{\mathbf{q}}+\vec{\mathbf{k}}-\vec{\mathbf{p}})](\vec{\mathbf{q}}+\vec{\mathbf{k}}-\vec{\mathbf{p}})^2}. \tag{44}$$

Needless to say, the equivalence between equations (43) and (44) can also be checked by direct computation. Since the argument of the trigonometric function depends on k_F and, therefore, on V, the exchange correlation energy is no longer a constant. As consequence, the thermodynamics properties are modified by the noncommutativity.

We have not been able to compute analytically the integral in equation (44). To proceed further, we assume that the global features of the system are insensitive to the direction of the vector $\theta^i \equiv 1/2\epsilon^{ijk}\Theta_E^{jk}$. We may, then, replace the right-hand side of equation (44) by its average over all possible directions of $\vec{\theta}$. This is effectively achieved by integrating over the angles of $\vec{\theta}$ which yields

$$\begin{aligned}\epsilon_0^{(2)b}(\theta) &\equiv \frac{1}{4\pi}\int \mathrm{d}\Omega_{\vec{\theta}}\epsilon_0^{(2)b}(\vec{\theta}) \\ &= \frac{3}{16\pi^5}\int\frac{\mathrm{d}^3\mathbf{q}}{\mathbf{q}^2}\int_{|\vec{\mathbf{k}}+\vec{\mathbf{q}}|>1}\mathrm{d}^3\mathbf{k}\int_{|\vec{\mathbf{p}}-\vec{\mathbf{q}}|>1}\mathrm{d}^3\mathbf{p}\frac{\xi(1-\mathbf{k})\xi(1-\mathbf{p})}{[\vec{\mathbf{q}}\cdot(\vec{\mathbf{q}}+\vec{\mathbf{k}}-\vec{\mathbf{p}})](\vec{\mathbf{q}}+\vec{\mathbf{k}}-\vec{\mathbf{p}})^2} \\ &\quad\times\frac{\sin(k_F^2\theta|\vec{\mathbf{q}}\times(\vec{\mathbf{k}}-\vec{\mathbf{p}})|)}{k_F^2\theta|\vec{\mathbf{q}}\times(\vec{\mathbf{k}}-\vec{\mathbf{p}})|}.\end{aligned} \tag{45}$$

Let us concentrate on analysing different limiting cases. For $\theta = |\vec{\theta}| = 0$ one returns unambiguously to the commutative model. On the other hand, when $\theta\to\infty \Longrightarrow \epsilon_0^{(2)b}\to 0$ implying that $\mathcal{E}_0^{(2)} < E_0^{(2)}$. However, thermodynamic quantities, such as pressure and bulk modulus, will remain unaltered because the difference $\mathcal{E}_0^{(2)} - E_0^{(2)}$ is just a constant.

The next step consists in bringing into play the Seiberg–Witten [4] map. By expanding the trigonometric function in equation (45) around $\theta = 0$, one arrives at

$$\epsilon_0^{(2)b}(\theta) = \epsilon_0^{(2)b} - \frac{1}{32\pi^5}k_F^4R\theta^2 + \mathcal{O}(\theta^4), \tag{46}$$

where

$$R = \int\frac{\mathrm{d}^3\mathbf{q}}{\mathbf{q}^2}\int_{|\vec{\mathbf{k}}+\vec{\mathbf{q}}|>1}\mathrm{d}^3\mathbf{k}\int_{|\vec{\mathbf{p}}-\vec{\mathbf{q}}|>1}\mathrm{d}^3\mathbf{p}\frac{\xi(1-\mathbf{k})\xi(1-\mathbf{p})|\vec{\mathbf{q}}\times(\vec{\mathbf{k}}-\vec{\mathbf{p}})|^2}{[\vec{\mathbf{q}}\cdot(\vec{\mathbf{q}}+\vec{\mathbf{k}}-\vec{\mathbf{p}})](\vec{\mathbf{q}}+\vec{\mathbf{k}}-\vec{\mathbf{p}})^2}. \tag{47}$$

The convergence of the $\mathbf{k}$ and $\mathbf{p}$ integrals is secured by the fact that they run over finite intervals. On the other hand, power counting tells us that the improper $\mathbf{q}$ integral also converges. Hence, R exists and is well defined. The situation changes drastically for those integrals that act as coefficients of higher orders in θ. There, power counting indicates that they are divergent. The way out of the trouble consists in carrying out the $\mathbf{q}$ integral between 0 and Λ, Λ being a cutoff such that, as $\theta\to 0$, $1/k_F^2\theta$ goes to infinity faster than Λ.

By collecting all the results, equation (37) yields

$$\begin{aligned}\mathcal{E}_0 &= \frac{e^2}{2a_0}N\left[\frac{2.21}{r_s^2} - \frac{0.916}{r_s} + 0.0622\ln r_s - 0.094 - \frac{1}{32\pi^5}k_F^4\theta^2R + \mathcal{O}(\theta^4, r_s\ln r_s)\right] \\ &= E_0 - N\frac{m}{\hbar}\frac{3}{32\pi^5}k_F^4e^2R\theta^2 + \mathcal{O}(\theta^4).\end{aligned} \tag{48}$$

The noncommutativity certainly modifies the ground state energy and, as consequence, the ionization potential of the material being treated as an electron gas. Moreover, from the comparison of equation (48) with the commutative electron gas at nonzero temperature [17], one may conclude that θ acts as a reference temperature [10] since $\partial\mathcal{E}_0/\partial\theta$ is a linear function of θ much as the specific heat at constant volume is a linear function of the absolute temperature. This is the main outcome of this paper, namely, the noncommutativity of the position observables lifts the degeneracy of the model and can be interpreted as if the electron gas would be at nonzero temperature.

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